# Problem 4.75

Consider the system of Example 4.6, now with a time-dependent flux $\Phi(t)$ through the solenoid. Show that

$$\Psi(t) = \frac{1}{\sqrt{2\pi}} e^{in\phi} e^{-if(t)}$$

with

$$f(t) = \frac{1}{\hbar} \int_0^t \frac{\hbar^2}{2mb^2} \left( n - \frac{q\Phi(t')}{2\pi\hbar} \right)^2 dt'$$

is a solution to the *time-dependent* Schrödinger equation.

[TYPO: Replace $\Psi(t)$ with $\Psi(\phi, t)$.]

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### Solution

The system of Example 4.6 consists of a particle with mass $m$ and charge $q$ that's constrained to move on a ring of radius $b$. There's a coaxial solenoid of radius $a$, where $a < b$, with an electric current running through it so that the vector potential outside the solenoid is

$$\mathbf{A} = \frac{\Phi(t)}{2\pi r} \hat{\boldsymbol{\phi}}, \quad r > a.$$

The governing equation for the wave function is Schrödinger's equation. Assuming the solenoid is uncharged, the scalar potential is $\varphi = 0$, so the Hamiltonian simplifies from its usual form for a particle in electromagnetic fields (see Problem 4.45).

$$\begin{aligned}
i\hbar \frac{\partial \Psi}{\partial t} &= \hat{H} \Psi \\
&= \left[ \frac{1}{2m} (\mathbf{p} - q\mathbf{A})^2 + q\varphi \right] \Psi \\
&= \frac{1}{2m} (-\hbar^2 \nabla^2 + q^2 A^2 + 2i\hbar q \mathbf{A} \cdot \nabla) \Psi \\
&= \frac{1}{2m} [-\hbar^2 \nabla^2 \Psi + q^2 (\mathbf{A} \cdot \mathbf{A}) \Psi + 2iq\hbar (\mathbf{A} \cdot \nabla \Psi)]
\end{aligned}$$

Since the particle is confined to move on a ring, expand the right side in cylindrical coordinates $(r, \phi, z)$, set $r = b$, and recognize that there's no variation in $\Psi$ as $r$ or $z$ change.

$$\begin{aligned}
i\hbar \frac{\partial \Psi}{\partial t} = \frac{1}{2m} \Bigg\{ -\hbar^2 \Bigg( \underbrace{\frac{\partial^2 \Psi}{\partial r^2}}_{=0} + \frac{1}{b} \underbrace{\frac{\partial \Psi}{\partial r}}_{=0} + \frac{1}{b^2} \frac{\partial^2 \Psi}{\partial \phi^2} + \underbrace{\frac{\partial^2 \Psi}{\partial z^2}}_{=0} \Bigg) + q^2 \left[ \frac{\Phi(t)}{2\pi b} \hat{\boldsymbol{\phi}} \right] \cdot \left[ \frac{\Phi(t)}{2\pi b} \hat{\boldsymbol{\phi}} \right] \Psi(\phi, t) \\
+ 2iq\hbar \left[ \frac{\Phi(t)}{2\pi b} \hat{\boldsymbol{\phi}} \right] \cdot \Bigg( \underbrace{\frac{\partial \Psi}{\partial r}}_{=0} \hat{\mathbf{r}} + \frac{1}{b} \frac{\partial \Psi}{\partial \phi} \hat{\boldsymbol{\phi}} + \underbrace{\frac{\partial \Psi}{\partial z}}_{=0} \hat{\mathbf{z}} \Bigg) \Bigg\}
\end{aligned}$$

$$= -\frac{\hbar^2}{2mb^2} \frac{\partial^2 \Psi}{\partial \phi^2} + \frac{iq\hbar}{2\pi mb^2} [\Phi(t)] \frac{\partial \Psi}{\partial \phi} + \frac{q^2}{8\pi^2 mb^2} [\Phi(t)]^2 \Psi(\phi, t), \quad 0 \le \phi \le 2\pi, \ t > 0 \qquad (1)$$

This partial differential equation is not separable despite being linear and homogeneous, so introduce the operator $L_z = -i\hbar(\partial/\partial\phi)$ in order to eliminate the partial derivatives in $\phi$. When $L_z$ acts on a wave function, the result is $\hbar m_\ell$ times that wave function. $m_\ell$ is the magnetic quantum number and takes on integer values: $m_\ell = 0, \pm1, \pm2, \ldots, \pm\ell$, where $\ell = 0, 1, 2, \ldots$.

$$\begin{aligned}
i\hbar\frac{\partial\Psi}{\partial t} &= \frac{1}{2mb^2}\left(-i\hbar\frac{\partial}{\partial\phi}\right)\left(-i\hbar\frac{\partial}{\partial\phi}\right)\Psi - \frac{q}{2\pi mb^2}[\Phi(t)]\left(-i\hbar\frac{\partial}{\partial\phi}\right)\Psi + \frac{q^2}{8\pi^2 mb^2}[\Phi(t)]^2\Psi(\phi,t) \\
&= \frac{1}{2mb^2}L_z^2\Psi - \frac{q}{2\pi mb^2}[\Phi(t)]L_z\Psi + \frac{q^2}{8\pi^2 mb^2}[\Phi(t)]^2\Psi(\phi,t) \\
&= \frac{1}{2mb^2}(\hbar m_\ell)^2\Psi - \frac{q}{2\pi mb^2}[\Phi(t)](\hbar m_\ell)\Psi + \frac{q^2}{8\pi^2 mb^2}[\Phi(t)]^2\Psi(\phi,t) \\
&= \left\{\frac{\hbar^2 m_\ell^2}{2mb^2} - \frac{q\hbar m_\ell}{2\pi mb^2}[\Phi(t)] + \frac{q^2}{8\pi^2 mb^2}[\Phi(t)]^2\right\}\Psi(\phi,t) \\
&= \frac{q^2}{8\pi^2 mb^2}\left\{[\Phi(t)]^2 - \frac{4m_\ell\pi\hbar}{q}[\Phi(t)] + \frac{4m_\ell^2\pi^2\hbar^2}{q^2}\right\}\Psi(\phi,t) \\
&= \frac{q^2}{8\pi^2 mb^2}\left(\Phi(t) - \frac{2m_\ell\pi\hbar}{q}\right)^2\Psi(\phi,t) \\
&= \frac{q^2}{8\pi^2 mb^2}\frac{4\pi^2\hbar^2}{q^2}\left(\frac{q\Phi(t)}{2\pi\hbar} - m_\ell\right)^2\Psi(\phi,t) \\
&= \frac{\hbar^2}{2mb^2}\left(m_\ell - \frac{q\Phi(t)}{2\pi\hbar}\right)^2\Psi(\phi,t)
\end{aligned}$$

Divide both sides by $i\hbar\Psi(\phi,t)$.

$$\frac{1}{\Psi(\phi,t)}\frac{\partial\Psi}{\partial t} = \frac{1}{i\hbar}\frac{\hbar^2}{2mb^2}\left(m_\ell - \frac{q\Phi(t)}{2\pi\hbar}\right)^2$$

Use the chain rule to rewrite the left side as the derivative of a logarithm.

$$\frac{\partial}{\partial t}\ln|\Psi(\phi,t)| = -\frac{i}{\hbar}\frac{\hbar^2}{2mb^2}\left(m_\ell - \frac{q\Phi(t)}{2\pi\hbar}\right)^2$$

An absolute value sign is placed around $\Psi(\phi,t)$ because the logarithm argument has to be positive. Note that $|\Psi(\phi,t)| = \sqrt{|\Psi(\phi,t)|^2} = \sqrt{\Psi^*\Psi}$. Integrate both sides partially with respect to time from 0 to $t$.

$$\int_0^t \frac{\partial}{\partial t'}\ln|\Psi(\phi,t')|\,dt' = -\frac{i}{\hbar}\int_0^t \frac{\hbar^2}{2mb^2}\left(m_\ell - \frac{q\Phi(t')}{2\pi\hbar}\right)^2 dt'$$

Relabel $m_\ell$ as $n$.

$$\ln|\Psi(\phi,t)| - \ln|\Psi(\phi,0)| = -i\left[\frac{1}{\hbar}\int_0^t \frac{\hbar^2}{2mb^2}\left(n - \frac{q\Phi(t')}{2\pi\hbar}\right)^2 dt'\right]$$

Combine the logarithms on the left and use $f(t)$ for the function in square brackets on the right.

$$\ln\left|\frac{\Psi(\phi,t)}{\Psi(\phi,0)}\right| = -if(t)$$

Exponentiate both sides.

$$\left|\frac{\Psi(\phi,t)}{\Psi(\phi,0)}\right| = e^{-if(t)}$$

Remove the absolute value sign by placing $\pm$ on the right side.

$$\frac{\Psi(\phi,t)}{\Psi(\phi,0)} = \pm e^{-if(t)}$$

Multiply both sides by $\Psi(\phi,0)$.

$$\Psi(\phi,t) = \pm\Psi(\phi,0)e^{-if(t)}$$

Use $C(\phi)$ for $\pm\Psi(\phi,0)$.

$$\Psi(\phi,t) = C(\phi)e^{-if(t)} \tag{2}$$

To determine $C(\phi)$, use the fact that $L_z\Psi(\phi,t) = n\hbar\Psi(\phi,t)$.

$$L_z\Psi(\phi,t) = n\hbar\Psi(\phi,t)$$

$$\left(-i\hbar\frac{\partial}{\partial\phi}\right)\Psi(\phi,t) = n\hbar\Psi(\phi,t)$$

$$-i\hbar\frac{\partial\Psi}{\partial\phi} = n\hbar\Psi(\phi,t)$$

$$-i\hbar\frac{\partial}{\partial\phi}[C(\phi)e^{-if(t)}] = n\hbar[C(\phi)e^{-if(t)}]$$

$$-i\hbar C'(\phi)e^{-if(t)} = n\hbar C(\phi)e^{-if(t)}$$

$$\frac{C'(\phi)}{C(\phi)} = in$$

$$\frac{d}{d\phi}\ln|C(\phi)| = in$$

$$\ln|C(\phi)| = in\phi + D$$

$$|C(\phi)| = e^{in\phi+D}$$

$$\begin{aligned} C(\phi) &= \pm e^{D}e^{in\phi} \\ &= C_1e^{in\phi} \end{aligned}$$

As a result, equation (2) becomes

$$\Psi(\phi, t) = C_1 e^{in\phi} e^{-if(t)}.$$

Determine $C_1$ by requiring the wave function to be normalized: Integrate $|\Psi(\phi, t)|^2$ over all possible values of $\phi$ and set the result equal to 1.

$$\begin{aligned}
1 = \int_0^{2\pi} |\Psi(\phi, t)|^2 \, d\phi = \int_0^{2\pi} \Psi^*(\phi, t)\Psi(\phi, t) \, d\phi &= \int_0^{2\pi} [C_1^* e^{-in\phi} e^{if(t)}][C_1 e^{in\phi} e^{-if(t)}] \, d\phi \\
&= |C_1|^2 \int_0^{2\pi} d\phi \\
&= |C_1|^2 (2\pi)
\end{aligned}$$

Solve for $C_1$, ignoring any phase factor.

$$C_1 = \pm \frac{1}{\sqrt{2\pi}}$$

Either the plus sign or the minus sign can be used because the probability density is $|\Psi(\phi, t)|^2$. Therefore, choosing the plus sign, the normalized wave function is

$$\boxed{\Psi(\phi, t) = \frac{1}{\sqrt{2\pi}} e^{in\phi} e^{-if(t)}.}$$

We can check that this is a solution by plugging it back into equation (1), the Schrödinger equation.

$$i\hbar \frac{\partial \Psi}{\partial t} \stackrel{?}{=} -\frac{\hbar^2}{2mb^2} \frac{\partial^2 \Psi}{\partial \phi^2} + \frac{iq\hbar}{2\pi mb^2} [\Phi(t)] \frac{\partial \Psi}{\partial \phi} + \frac{q^2}{8\pi^2 mb^2} [\Phi(t)]^2 \Psi(\phi, t) \tag{1}$$

$$\begin{aligned}
i\hbar \frac{\partial}{\partial t} \left( \frac{1}{\sqrt{2\pi}} e^{in\phi} e^{-if(t)} \right) \stackrel{?}{=} -\frac{\hbar^2}{2mb^2} \frac{\partial^2}{\partial \phi^2} \left( \frac{1}{\sqrt{2\pi}} e^{in\phi} e^{-if(t)} \right) + \frac{iq\hbar}{2\pi mb^2} [\Phi(t)] \frac{\partial}{\partial \phi} \left( \frac{1}{\sqrt{2\pi}} e^{in\phi} e^{-if(t)} \right) \\
+ \frac{q^2}{8\pi^2 mb^2} [\Phi(t)]^2 \left( \frac{1}{\sqrt{2\pi}} e^{in\phi} e^{-if(t)} \right)
\end{aligned}$$

$$\begin{aligned}
i\hbar \frac{-i}{\sqrt{2\pi}} e^{in\phi} e^{-if(t)} f'(t) \stackrel{?}{=} -\frac{\hbar^2}{2mb^2} \frac{(in)^2}{\sqrt{2\pi}} e^{in\phi} e^{-if(t)} + \frac{iq\hbar}{2\pi mb^2} [\Phi(t)] \frac{(in)}{\sqrt{2\pi}} e^{in\phi} e^{-if(t)} \\
+ \frac{q^2}{8\pi^2 mb^2} [\Phi(t)]^2 \left( \frac{1}{\sqrt{2\pi}} e^{in\phi} e^{-if(t)} \right)
\end{aligned}$$

$$\hbar f'(t) \stackrel{?}{=} -\frac{\hbar^2}{2mb^2} (in)^2 + \frac{iq\hbar}{2\pi mb^2} [\Phi(t)](in) + \frac{q^2}{8\pi^2 mb^2} [\Phi(t)]^2$$

Substitute $f(t)$ and use the fundamental theorem of calculus on the left.

$$\hbar\frac{d}{dt}\left[\frac{1}{\hbar}\int_0^t \frac{\hbar^2}{2mb^2}\left(n-\frac{q\Phi(t')}{2\pi\hbar}\right)^2 dt'\right] \stackrel{?}{=} -\frac{\hbar^2}{2mb^2}(in)^2 + \frac{iq\hbar}{2\pi mb^2}[\Phi(t)](in) + \frac{q^2}{8\pi^2 mb^2}[\Phi(t)]^2$$

$$\frac{\hbar^2}{2mb^2}\frac{d}{dt}\int_0^t \left(n-\frac{q\Phi(t')}{2\pi\hbar}\right)^2 dt' \stackrel{?}{=} \frac{n^2\hbar^2}{2mb^2} - \frac{nq\hbar}{2\pi mb^2}[\Phi(t)] + \frac{q^2}{8\pi^2 mb^2}[\Phi(t)]^2$$

$$\left(n-\frac{q\Phi(t)}{2\pi\hbar}\right)^2 \stackrel{?}{=} n^2 - \frac{nq}{\pi\hbar}[\Phi(t)] + \frac{q^2}{4\pi^2\hbar^2}[\Phi(t)]^2$$

$$n^2 - \frac{nq}{\pi\hbar}[\Phi(t)] + \frac{q^2}{4\pi^2\hbar^2}[\Phi(t)]^2 = n^2 - \frac{nq}{\pi\hbar}[\Phi(t)] + \frac{q^2}{4\pi^2\hbar^2}[\Phi(t)]^2$$

Consequently, the formula for $\Psi(\phi, t)$ satisfies the Schrödinger equation. Note that it works for any value of $n$, not just integers.